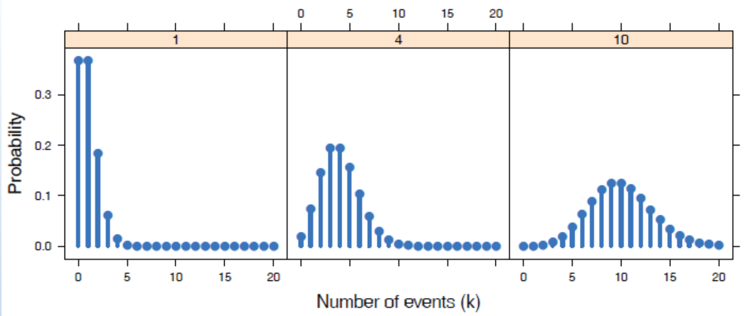

Discrete distributions

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Discrete distributions: Basic ideas

- Quantitative data: often assumed Normal (μ, σ^2) – unreasonable for CDA
- Binomial, Poisson, Negative binomial, … are the building blocks for CDA
- Form the basis for modeling techniques
  - logistic regression, generalized linear models, Poisson regression
- Data:
  - outcome variable (k = 0, 1, 2, …)
  - counts of occurrences (n_k): accidents, words in text, males in families of size k

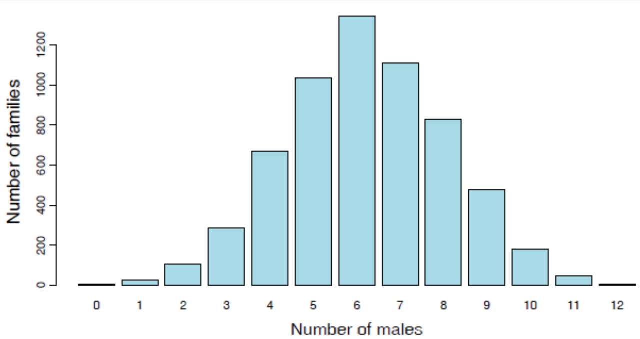
Examples: binomial

Human sex ratio (Geissler, 1889): Is there evidence that Pr(male) = 0.5?

Saxony families

Saxony families with 12 children having $k = 0, 1, \ldots 12$ sons.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
n_k	3	24	104	286	670	1033	1343	1112	829	478	181	45	7



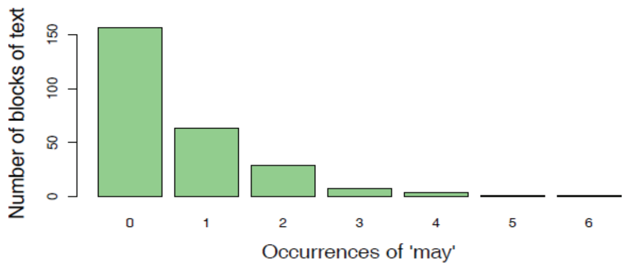
Examples: count data

Federalist papers: Disputed authorship

- 77 essays by Alexander Hamilton, John Jay, James Madison to persuade voters to ratify the US constitution, all signed with pseudonym “Publius”
  - Who wrote each?
  - 65 known, 12 disputed (H & M both claimed sole authorship)
- Mosteller & Wallace (1984): analysis of frequency distns of key “marker” words: from, **may**, whilst, …
- e.g., blocks of 200 words: occurrences (k) of “may” in how many blocks (n_k)

```
> data(Federalist, package = "vcd")
> Federalist
nMay
  0   1   2   3   4   5   6
156  63  29   8   4   1   1
```

Count data: models



For each word (“from”, “may”, “whilst”, …)

- Fit a probability model (Poisson, NegBin)
- Estimate parameters (λ, θ)
- $\rightarrow$ Calculate log Odds (Hamilton vs. Madison)
- $\rightarrow$ All 12 disputed papers most likely written by Madison

Example: Type-token distributions

- Basic count, k: number of “types”; frequency, n_k: number of instances observed
    - Frequencies of distinct words in a book or literary corpus
    - Number of subjects listing words as members of the semantic category “fruit”
    - Distinct species of animals caught in traps
- Differs from other distributions in that the frequency for $k = 0$ is *unobserved*
- Distribution is often extremely skewed (J-shaped)

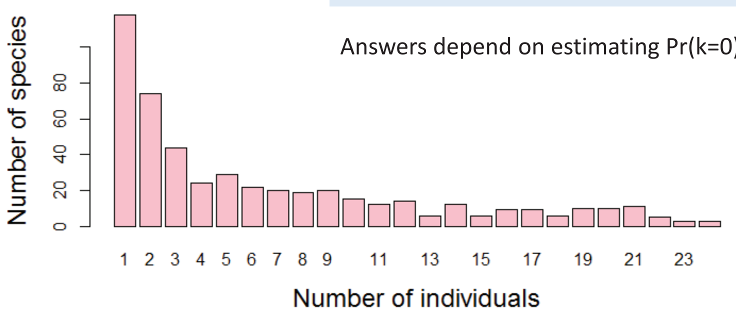
Table: Number of butterfly species n_k for which k individuals were collected

Individuals (k)	1	2	3	4	5	6	7	8	9	10	11	12	
Species (n_k)	118	74	44	24	29	22	20	19	20	15	12	14	
Individuals (k)	13	14	15	16	17	18	19	20	21	22	23	24	Su
Species (n_k)	6	12	6	9	9	6	10	10	11	5	3	3	5


```
data(Butterfly, package="vcd")
barplot(Butterfly,
        xlab = "Number of individuals",
        ylab = "Number of species",
        col = "pink",
        cex.lab = 1.5)
```

Questions:
What is the total pop. of butterflies in Malaysia?
How many wolves remain in Canada NWT?
How many words did Shakespeare know?

Answers depend on estimating Pr(k=0)



Discrete distributions: Questions

- General questions
    - What process gave rise to the distribution?
    - What is the form: uniform, binomial, Poisson, negative binomial, … ?
    - $\rightarrow$ Fit & estimate parameters
        - Visualize goodness of fit
    - $\rightarrow$ Use in some larger context to tell a story
- Examples
    - *Families in Saxony*: might expect Bin(n=12, p); p=0.5?
    - *Federalist papers*: Perhaps Poisson(λ)
    - *Butterfly data*: Perhaps a log-series distribution?

Fitting discrete distributions

Lack of fit:

- Lack of fit tells us something about the process giving rise to the data
- Poisson: assumes constant small probability of the basic event
- Binomial: assumes constant probability and independent trials
- Negative binomial: allows for *overdispersion*, relative to Poisson

Motivation:

- Models for more complex categorical data use these basic discrete distributions
- Binomial (with predictors) $\rightarrow$ logistic regression
- Poisson (with predictors) $\rightarrow$ poisson regression, loglinear models
- $\Rightarrow$ many of these are special cases of *generalized linear models*

Common discrete distributions

Discrete distributions are characterized by a probability function, Pr(X = k) ≡ p(k), that the random variable X has value k.

- Common discrete distributions have the following forms:

Discrete distribution	Probability function, $p(k)$	Parameters
Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$	p = Pr (success); n = # trials
Poisson	$e^{-\lambda}\lambda^k / k!$	λ = mean
Negative binomial	$\binom{n+k-1}{k} p^n (1-p)^k$	p; n = # *successful* trials
Geometric	$p(1-p)^k$	p
Logarithmic series	$\theta^k / [-k \log(1-\theta)]$	θ

Discrete distributions: R

R functions: {d, p, q, r}

- `d____` density function, Pr(X=k) = p(k)
- `p____` cumulative probability, F(k) = $\sum_{X \le k} p(k)$
- `q____` quantile function, k = F^{-1} (p), smallest value such that $F(k) \ge p$
- `r____` random number generator

Discrete distribution	Density (pmf) function	Cumulative (CDF)	Quantile CDF^{-1}	Random # generator
Binomial	`dbinom()`	`pbinom()`	`qbinom()`	`rbinom()`
Poisson	`dpois()`	`ppois()`	`qpois()`	`rpois()`
Negative binomial	`dnbinom()`	`pnbinom()`	`qnbinom()`	`rnbinom()`
Geometric	`dgeom()`	`pgeom()`	`qgeom()`	`rgeom()`
Logarithmic series	`dlogseries()`	`plogseries()`	`qlogseries()`	`rlogseries()`

Binomial distribution

The binomial distribution, $\text{Bin}(n, p)$,

ways to get k out of n   Pr(k events)   Pr(n-k non-events)

$$\text{Bin}(n,p) : \Pr\{X = k\} \equiv p(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \ldots, n\ , \qquad (1)$$

arises as the distribution of the number of events of interest ("successes") which occur in n *independent trials* when the probability of the event on any one trial is the *constant* value $p = \Pr(\text{event})$.

Examples

- Toss 10 fair coins– how many heads? Bin(10, ½)
- Toss 12 fair dice– how many 5s or 6s? Bin(12, 1/3)

Mean, variance, skewness:

Mean[X] = n p

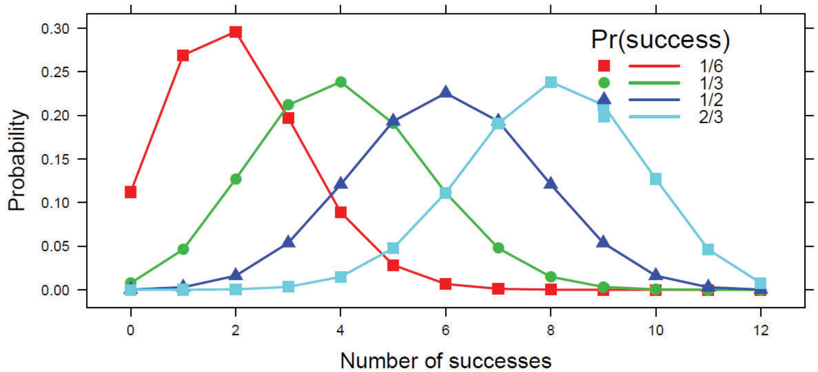
Var[X] = n p (1-p) = n p q

Skew[X] = n p q (q-p)

MLE from data: $\hat{p} = \frac{\bar{x}}{n} = \frac{\sum_k k \times n_k / \sum_k n_k}{n}$

Binomial distribution

Binomial distributions for k = 0, 1, 2, …, 12 successes in n=12 trials, for 4 values of p



- Mean = n p
- Variance is maximum when p = ½
- Skewed when p ≠ ½

DDAR Fig 3.9, pp 76-77

Poisson distribution

The Poisson distribution, Pois(λ),

$$\text{Pois}(\lambda) : \Pr\{X = k\} \equiv p(k) = \frac{e^{-\lambda}\,\lambda^k}{k!} \qquad k = 0, 1, \ldots \qquad (2)$$

gives the probability of an event occurring $k = 0, 1, 2, \ldots$ times over a *large number of independent* trials, when the probability, p, that the event occurs on any one trial (in time or space) is *small and constant*.

Examples:

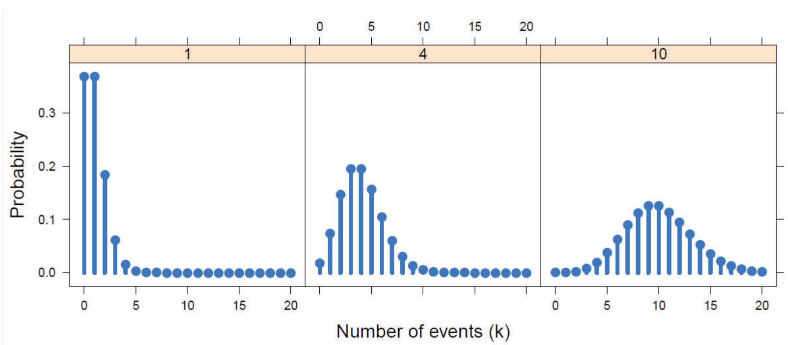
- Number of highway accidents at some given location
- Defects in a manufacturing process
- Number of goals scored in soccer games

Table: Total goals scored in 380 games in the Premier Football League, 1995/95 season

Total goals	0	1	2	3	4	5	6	7
Number of games	27	88	91	73	49	31	18	3

Poisson distribution

Poisson distributions for λ = 1, 4, 10



DDAR Fig 3.10, p 81

Mean, variance, skewness:

Mean[X] = λ

Var[X] = λ

Skew[X] = $\lambda^{-1/2}$

MLE: $\hat{\lambda} = \bar{x}$

Properties:

Sum of Pois (λ_1, λ_2, λ_3, …) = Pois($\sum\lambda_i$)

Approaches N(λ, λ) as n $\rightarrow \infty$

Negative binomial distribution

The Negative binomial distribution, NBin(n, p),

$$\text{NBin}(n, p) : \Pr\{X = k\} \equiv p(k) = \binom{n+k-1}{k} p^n (1-p)^k \qquad k = 0, 1, \ldots, \infty$$

is a waiting time distribution. It arises when n trials are observed with constant probability p of some event, and we ask how many non-events (failures), k, it takes to observe n successful events.

Example: Toss a coin; what is probability of getting $k = 0, 1, 2, \ldots$ tails before $n = 3$ heads?

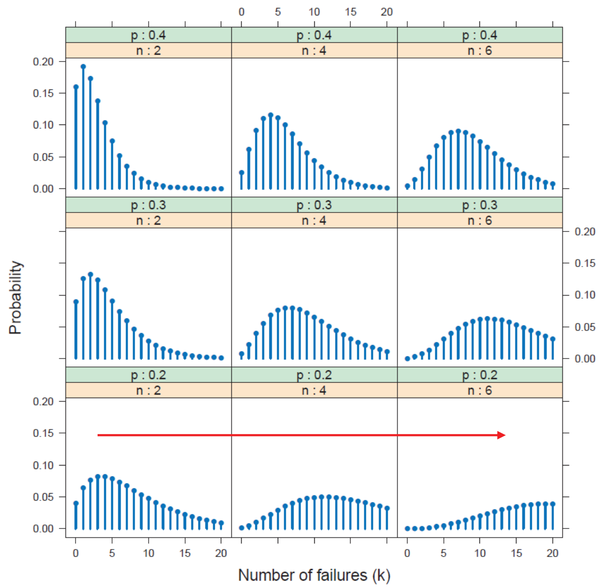
This distribution is often used as an alternative to the Poisson when

- constant probability p or independence are violated
- variance is greater than the mean (overdispersion: Var[X] $>$ Mean[X])

$$\begin{aligned} \text{Mean}(X) &= nq/p = \mu \\ \text{Var}(X) &= nq/p^2 \\ \text{Skew}(X) &= \frac{2-p}{\sqrt{nq}}, \end{aligned}$$

$$\text{Mean}(X) = \mu = \frac{n(1-p)}{p} \implies p = \frac{n}{n+\mu},$$

$$\text{Var}(X) = \frac{n(1-p)}{p^2} \implies \text{Var}(X) = \mu + \frac{\mu^2}{n}.$$



Negative binomial distributions for
n = 2, 4, 6
p = 0.2, 0.3, 0.4

Mean:
Increases with n
Decreases with p

DDAR Fig 3.13, p 85

Fitting discrete distributions

Fitting a discrete distribution involves the following steps:

1. Estimate the parameter(s) from the data, e.g., p for binomial, λ for Poisson, etc. Typically done using maximum likelihood, but some distributions have simple expressions:
   - Binomial, $\hat{p} = \sum k n_k / (n \sum n_k)$ = mean / n
   - Poisson, $\hat{\lambda} = \sum k n_k / \sum n_k$ = mean
2. Calculate fitted probabilities, $\hat{p}(k)$ for the distribution, and then fitted frequencies, $N\hat{p}(k)$.
3. Assess Goodness of fit: Pearson X^2 or likelihood-ratio G^2

$$X^2 = \sum_{k=1}^{K} \frac{(n_k - N\hat{p}_k)^2}{N\hat{p}_k} \qquad G^2 = \sum_{k=1}^{K} n_k \log(\frac{n_k}{N\hat{p}_k})$$

Both have asymptotic chisquare distributions, χ^2_{K-s} with s estimated parameters, under the hypothesis that the data follows the chosen distribution.

Fitting & graphing discrete distributions

In R, the vcd and vcdExtra packages provide functions to fit, visualize and diagnose discrete distributions

- **Fitting**: `goodfit()` fits uniform, binomial, Poisson, neg bin, geometric, logseries, …
- **Graphing**: `rootogram()` assess departure between observed, fitted counts
- **Ord plot**: `Ordplot()` diagnose form of a discrete distribution
- **Robust plots**: `distplot()` handle problems with discrepant counts

Example: Saxony families

```
> data(Saxony, package="vcd")
> Saxony
nMales
   0    1    2    3    4    5    6    7    8    9   10   11   12
   3   24  104  286  670 1033 1343 1112  829  478  181   45    7
```

Use goodfit() to fit the binomial; test with summary()

```
> Sax.fit <- goodfit(Saxony, type = "binomial", par=list(size=12))
> summary(Sax.fit)

         Goodness-of-fit test for binomial distribution

                X^2 df P(> X^2)
Likelihood Ratio  97 11 6.98e-16
```

Example: Saxony families

The print() method for goodfit objects shows the details

```
> Sax.fit    # print

Observed and fitted values for binomial distribution
with parameters estimated by `ML'

 count observed   fitted pearson residual
     0        3    0.933            2.140
     1       24   12.089            3.426
     2      104   71.803            3.800
     3      286  258.475            1.712
     4      670  628.055            1.674
     5     1033 1085.211           -1.585
     6     1343 1367.279           -0.657
     7     1112 1265.630           -4.318
     8      829  854.247           -0.864
     9      478  410.013            3.358
    10      181  132.836            4.179
    11       45   26.082            3.704
    12        7    2.347            3.037
```

Pay attention to the pattern & magnitudes of residuals, d_k

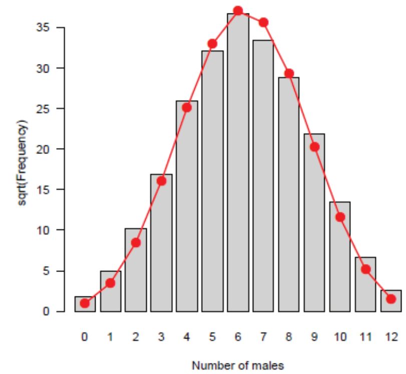
Pearson $\chi^2 = \sum d_k{}^2$

What's wrong with simple histograms?

Discrete distributions are often graphed as histograms, with a theoretical fitted distribution superimposed

The plot() method for goodfit objects provides some alternatives

```
> plot(Sax.fit, type = "standing", xlab = "Number of males")
```

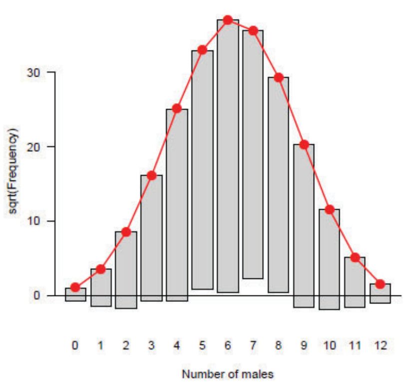


Problems:

- Largest frequencies dominate
- Must assess deviations vs. the fitted curve

Hanging rootograms

```
> plot(Sax.fit, type = "hanging", xlab = "Number of males")        # default
```



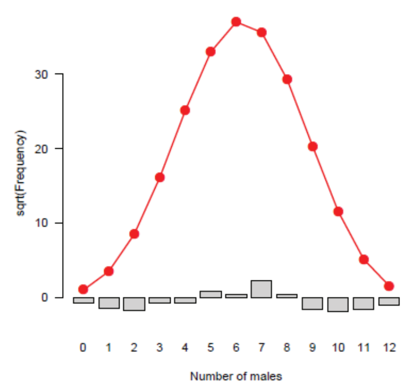
Tukey (1972, 1977):

- shift histogram bars to the fitted curve
- $\rightarrow$ judge deviations vs. horizontal line.
- plot $\sqrt{\text{freq}} \rightarrow$ smaller frequencies are emphasized.

We can now see clearly where the binomial doesn't fit

Deviation rootograms

```
> plot(Sax.fit, type = "deviation", xlab = "Number of males")
```



Deviation rootogram:

- emphasize differences between observed and fitted frequencies
- bars now show the residuals (gaps) directly

There are more families with very low or very high number of sons than the binomial predicts.

Q: Why is this so much better than the lack-of-fit test?

Example: Federalist papers

```
> data(Federalist, package="vcd")
> Federalist
nMay
  0    1    2    3    4    5    6
156   63   29    8    4    1    1
```

Fit the Poisson distribution

```
> Fed.fit0 <- goodfit(Federalist, type="poisson")
> summary(Fed.fit0)

         Goodness-of-fit test for poisson
distribution

                   X^2 df P(> X^2)
Likelihood Ratio 25.2  5 0.000125
```

This fits very poorly!

Example: Federalist papers

Try the Negative binomial distribution

```
> Fed.fit1<- goodfit(Federalist, type="nbinomial")
> summary(Fed.fit1)

        Goodness-of-fit test for nbinomial distribution

                  X^2 df P(> X^2)
Likelihood Ratio 1.96  4    0.742
```

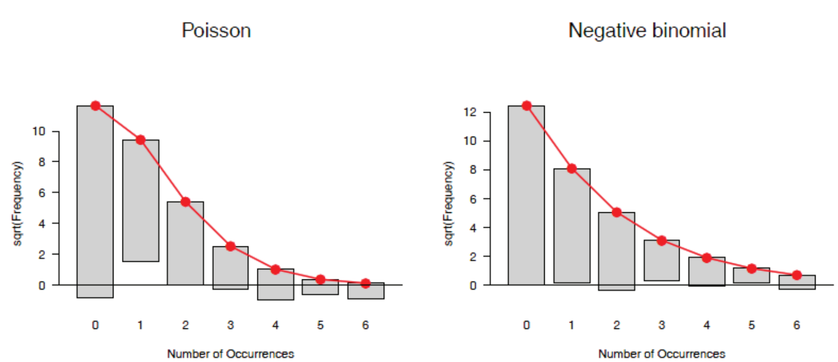
This now fits very well, indeed! Why?

- Poisson assumes that the probability of a given word ("may") is constant across all blocks of text.
- Negative binomial allows the rate parameter λ to vary over blocks of text

Federalist papers: Rootograms

Hanging rootograms for the Federalist papers data, comparing Poisson and Negative binomial

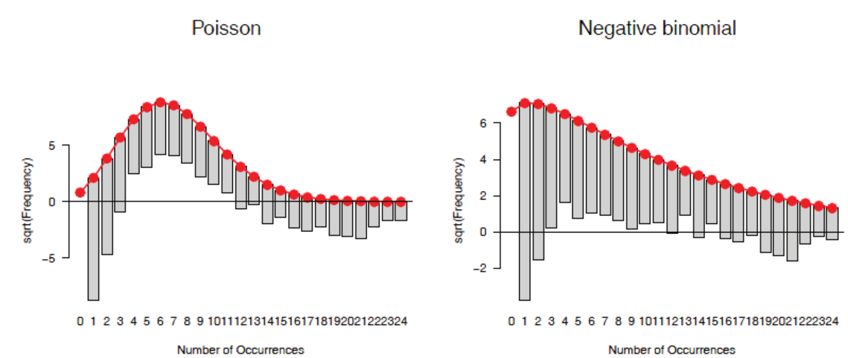
```
> plot(Fed.fit0, main = "Poisson")
> plot(Fed.fit1, main = "Negative binomial")
```



Butterfly data

Both Poisson and Negative binomial are terrible fits!   What to do??

```
But.fit1 <- goodfit(Butterfly, type="poisson")
But.fit2 <- goodfit(Butterfly, type="nbinomial")
plot(But.fit1, main="Poisson")
plot(But.fit2, main="Negative binomial")
```



Ord plots: Diagnose form of distribution

How to tell which discrete distributions are likely candidates?

- Ord (1967): for each of Poisson, Binomial, Negative binomial, and Logarithmic series distributions,
  - plot of kp_k/p_{k-1} against k is linear
  - signs of intercept and slope $\rightarrow$ determine the form, give rough estimates of parameters

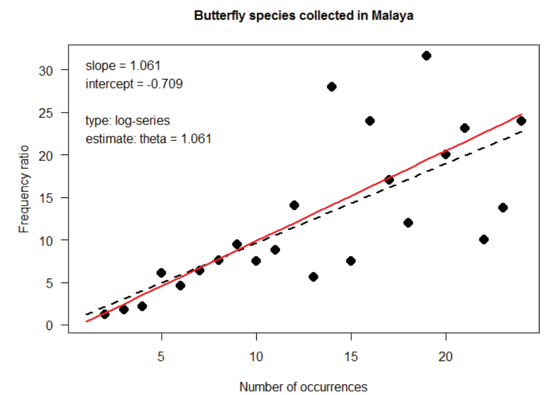
Slope (b)	Intercept (a)	Distribution (parameter)	Parameter estimate
0	+	Poisson (λ)	$\lambda = a$
−	+	Binomial (n, p)	$p = b/(b-1)$
+	+	Neg. binomial (n,p)	$p = 1 - b$
+	−	Log. series (θ)	$\theta = b$
			$\theta = -a$

- Fit line by WLS, using $\sqrt{n_k - 1}$ as weights
- A heuristic method: doesn't always work, but often a good start.

Ord plot: Examples

Butterfly data: The slope and intercept correctly diagnoses the log-series distribution

```
> Ord_plot(Butterfly,
        main = "Butterfly species collected in Malaya",
        gp=gpar(cex=1), pch=16)
```

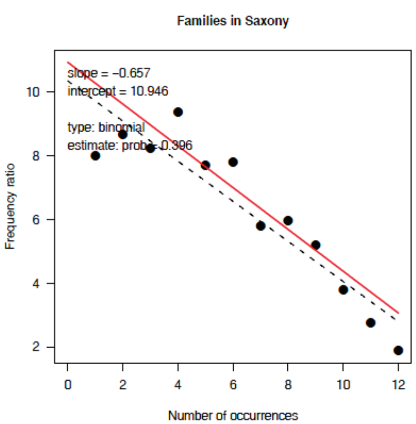


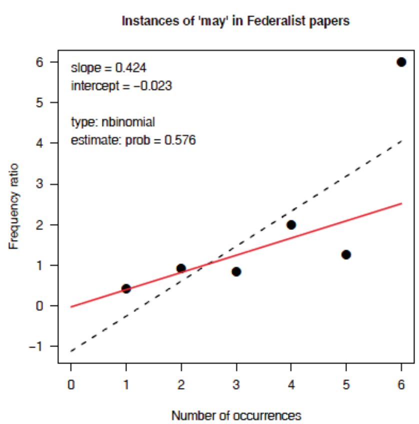
\+ slope
\- intercept
$\rightarrow$ log-series

Ord plots: Examples

Ord plots for the Saxony and Federalist data

```
> Ord_plot(Saxony, main = "Families in Saxony", gp=gpar(cex=1), pch=16)
> Ord_plot(Federalist, main = "Instances of 'may' in Federalist papers", gp=gpar(cex=1), pch=16)
```

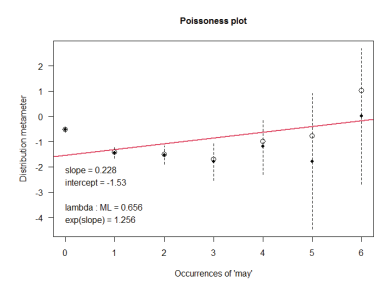




Robust distribution plots

- Ord plots lack robustness
  - one discrepant freqency, n_k affects points for both k and $k+1$
  - the use of WLS to fit the line is a small attempt to minimize this
- Robust plots for Poisson distribution (Hoaglin and Tukey, 1985)
  - For Poisson, plot ***count metameter*** $= \phi(n_k) = \log_e(k!\, n_k/N)$ vs. k
  - Linear relation $\Rightarrow$ Poisson, slope gives $\hat{\lambda}$
  - CI for points, diagnostic (influence) plot
  - Implemented in `distplot()` in the vcd package

For the Poisson distribution, this is called a "poissonness plot"



Poissonness plot: Details

- If the distribution of n_k is Poisson(λ) for some fixed λ, then each observed frequency, $n_k \approx m_k = Np_k$.
- Then, setting $n_k = Np_k = e^{-\lambda} \lambda^k / k!$, and taking logs of both sides gives

$$\log(n_k) = \log N - \lambda + k \log \lambda - \log k!$$

which can be rearranged to

$$\phi(n_k) \equiv \log\left(\frac{k!\, n_k}{N}\right) = -\lambda + (\log \lambda)\, k$$

- $\Rightarrow$ if the distribution is Poisson, plotting $\phi(n_k)$ vs. k should give a line with
  - intercept = $-\lambda$
  - slope = $\log \lambda$
- Nonlinear relation $\rightarrow$ distribution is *not* Poisson
- Hoaglin and Tukey (1985) give details on calculation of confidence intervals and influence measures.

Other distributions

This idea extends readily to other discrete data distributions:

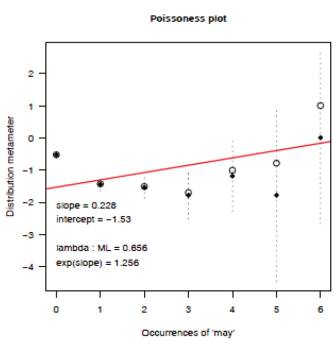
- The binomial, Poisson, negative binomial, geometric and logseries distributions are all members of a general power series family of discrete distributions. See: *DDAR*, Table 3.10 for details.
- This allows all of these to be represented in a plot of a suitable count metameter, $\phi(n_k)$ vs. k. See: *DDAR*, Table 3.12 for details.

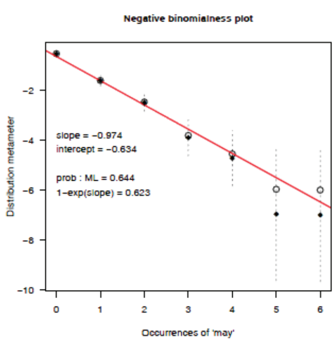
- In these plots, a straight line confirms that the data follow the given distribution.
- Confidence intervals around the points indicate uncertainty for the count metameter.
- The slope and intercept of the line give estimates of the distribution parameters.

distplot: Federalist

Try both Poisson & Negative binomial

```
distplot(Federalist, type="poisson", xlab="Occurrences of 'may'")
distplot(Federalist, type="nbinomial", xlab="Occurrences of 'may'")
```



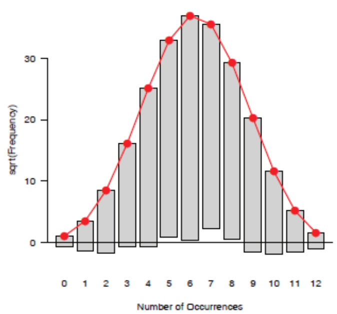


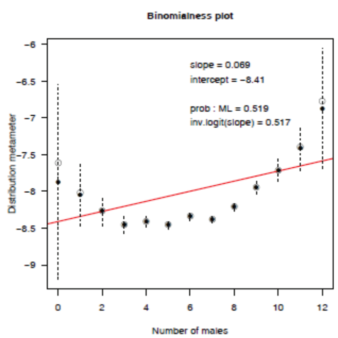
Again, the Poisson distribution is seen not to fit, while the Negative binomial appears reasonable.

distplot: Saxony

For purported binomial distributions, the result is a “binomialness” plot

```
plot(goodfit(Saxony, type="binomial", par=list(size=12)))
distplot(Saxony, type="binomial", size=12, xlab="Number of males")
```





Both plots show heavier tails than the binomial distribution. distplot() is more sensitive in diagnosing this

What have we learned?

Main points:

- Discrete distributions involve basic *counts* of occurrences of some event occurring with varying *frequency*.
- The ideas and methods for one-way tables are building blocks for analysis of more complex data.
- Commonly used discrete distributions include the binomial, Poisson, negative binomial, and logarithmic series distributions, all members of a *power series* family.
- Fitting observed data to a distribution $\rightarrow$ fitted frequencies, $N\hat{p}_k$, $\rightarrow$ goodness-of-fit tests (Pearson X^2, LR G^2)
- R: `goodfit()` provides `print()`, `summary()` and `plot()` methods.
- Plotting with rootograms, Ord plots and generalized distribution plots can reveal *how* or *where* a distribution does not fit.

What have we learned?

Some explantions:

- The Saxony data were part of a much larger data set from Geissler (1889) (`Geissler` in vcdExtra).
  - For the binomial, with families of size $n = 12$, our analyses give $\hat{p} = \Pr(male) = 0.52$.
  - Other analyses (using more complex models) conclude that p varies among families with the same size.
  - One explanation is that family decisions to have another child are influenced by the boy–girl ratio in earlier children.
- As suggested earlier, the lack of fit of the Poisson distribution for words in the Federalist papers can be explained by *context* of the writing:
  - Given "marker" words appear more or less often over time and subject than predicted by constant rates (λ) for a given author (Madison or Hamilton)
  - The negative binomial distribution fit much better.
  - The estimated parameters for these texts allowed assigning all 12 disputed papers to Madison.

Looking ahead: PhdPubs data

Example 3.24 in DDAR gives data on the number of publications by PhD candidates in the last 3 years of study

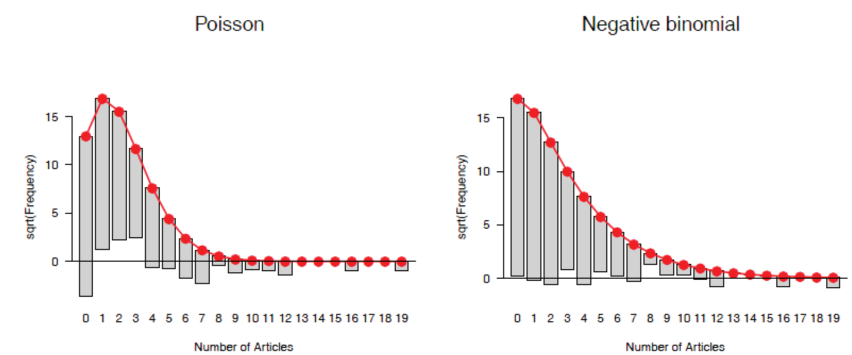
```
data("PhdPubs", package = "vcdExtra")
table(PhdPubs$articles)

##
##   0   1   2   3   4   5   6   7   8   9  10  11  12  16  19
## 275 246 178  84  67  27  17  12   1   2   1   1   2   1   1
```

- There are predictors: gender, marital status, number of children, prestige of dept., # pubs by student's mentor
- We fit such models with glm(), but need to specify the form of the distribution
- Ignoring the predictors for now, a baseline model could be

  glm(articles ~ 1, data=PhdPubs, family = "poisson")

Looking ahead: PhdPubs

```
plot(goodfit(PhdPubs$articles), xlab = "Number of Articles",
     main = "Poisson")
plot(goodfit(PhdPubs$articles, type = "nbinomial"),
     xlab = "Number of Articles", main = "Negative binomial")
```



Poisson doesn't fit: Need to account for excess 0s (some never published)
Neg binomial: Sort of OK, but should take predictors into account

Looking ahead: Count data models

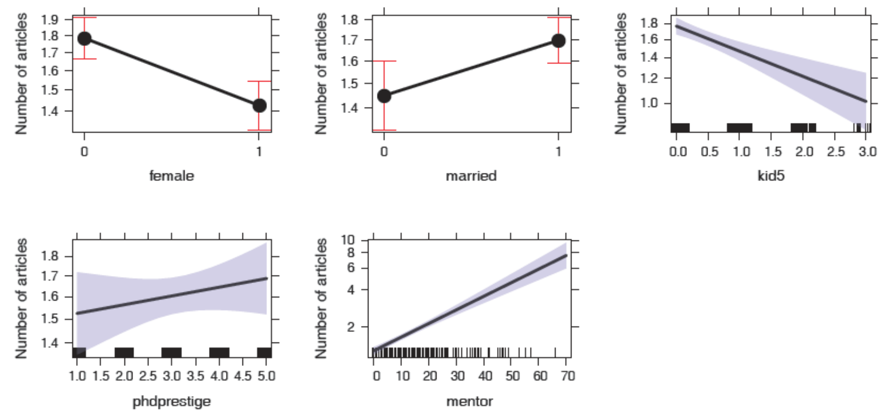
Count data regression models (DDAR Ch 11)

- Include predictors
- Allow different distributions for unexplained variation
- Provide tests of one model vs. another
- Special models handle the problems of excess zeros: zeroinlf(), hurdle()

```
# predictors: female, married, kid5, phdprestige, mentor
phd.pois  <- glm(articles ~ ., data=PhdPubs, family=poisson)
phd.nbin  <- glm.nb(articles ~ ., data=PhdPubs)

LRstats(phd.pois, phd.nbin)

## Likelihood summary table:
##           AIC  BIC LR Chisq  Df Pr(>Chisq)
## phd.pois 3313 3342     1634 909    <2e-16 ***
## phd.nbin 3135 3169     1004 909     0.015 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```


Looking ahead: Effect plots

Effect plots show the predicted values for each term in a model, averaging over all other factors.



These are better visual summaries for a model than a table of coefficients.

Summary

- Discrete distributions are the building blocks for categorical data analysis
  - Typically consist of basic counts of occurrences, with varying frequencies
  - Most common: binomial, Poisson, negative binomial
  - Others: geometric, log-series
- Fit with goodfit(); plot with rootogram()
  - Diagnostic plots: Ord_plot(), distplot()
- Models with predictors
  - Binomial → logistic regression
  - Poisson → poisson regression; logliner models
  - These are special cases of generalized linear models